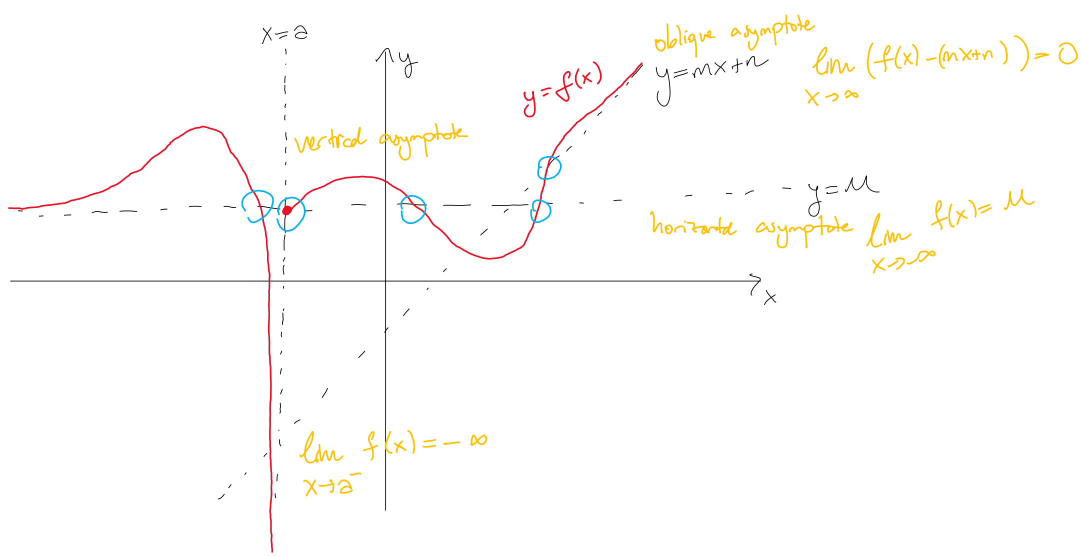
Defn: The graph $y=f(x)$ of a function f is said to have

- a vertical asymptote at $x=a$ if $\lim_{x\to a^+} f(x) = \mp\infty$ or $\lim_{x\to a^-} f(x) = \mp\infty$
- a horizontal asymptote $y=M$ if $\lim_{x\to\infty} f(x) = M$ or $\lim_{x\to-\infty} f(x) = M$
- an oblique asymptote $y=mx+n$ if $\lim_{x\to\infty} (f(x)-(mx+n))=0$ or $\lim_{x\to-\infty} (f(x)-(mx+n))=0$



How do we sketch the graph of a function?

- Find f' and f'' and the intervals of increase/decrease and the concavity of the function (Presumably, you'll need to use the first and second derivative tests.)
- Find the asymptotes, if there are any.
- Plug in some sample points (for example, you may the x- and y-intercepts.)

Example: Sketch the graph of $f(x) = x - 2\arctan(x)$.

Solution: We have that $f'(x) = 1 - 2\cdot\frac{1}{1+x^2} = \frac{x^2+1-2}{1+x^2} = \frac{(x-1)(x+1)}{1+x^2}$ and

$f''(x) = 0 - 2\cdot\frac{-1}{(1+x^2)^2}\cdot 2x = \frac{4x}{(1+x^2)^2}$. So $f'(x)=0 \Rightarrow x=1$ or -1

$f''(x) = 0 \Rightarrow x=0$

x	$-\infty$		-1		0		1		∞
f'		$+$	0	$-$			0	$+$	
f''				$-$	0	$+$			
f		↗ ⌢		↘ ⌢		↘ ⌣		↗ ⌣	
			local max		inflection point		local min		

f has no vertical asymptotes (as it is continuous everywhere)

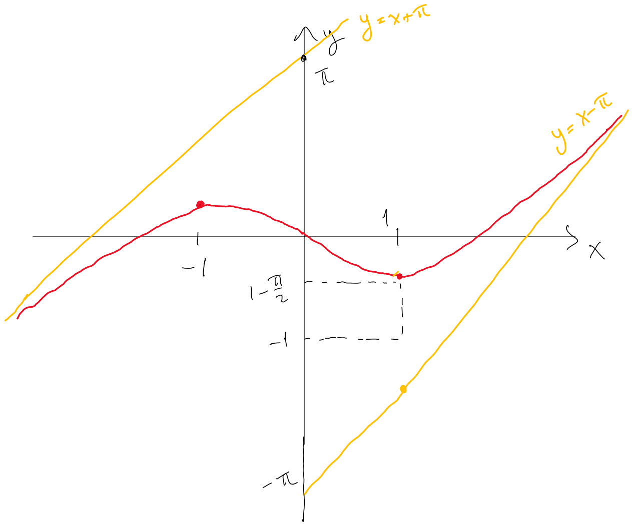
$\lim_{x\to\infty} x - 2\arctan(x) = +\infty$ (with $\arctan(x) \to \pi/2$) and $\lim_{x\to-\infty} x - 2\arctan(x) = -\infty$

So there are no horizontal asymptotes.

But $\lim_{x\to\infty} \left(\cancel{x} - 2\arctan(x) - (\cancel{x} - \pi)\right) = 0$ and

$\lim_{x\to-\infty} \left(\cancel{x} - 2\arctan(x) - (\cancel{x} + \pi)\right) = 0$. So we have two

oblique asymptotes.



$f(-1) = (-1) - 2\arctan(-1) = -1 - 2\cdot\left(\frac{-\pi}{4}\right) = \frac{\pi}{2} - 1$

$f(0) = 0 - 2\arctan(0) = 0$

$f(1) = 1 - \frac{\pi}{2}$

Example: Let $f(x) = \dfrac{x^2-1}{x^2-4}$. You are given that $f'(x) = \dfrac{-6x}{(x^2-4)^2}$ and $f''(x) = \dfrac{6(3x^2+4)}{(x^2-4)^3}$

Sketch the graph of f.

Solution:

x	$-\infty$	-2	0	2	$+\infty$
f'	+	+	0 −	−	
f''	+	−	−	+	
f	↗ ∪	↗ ∩	↘ ∩	↘ ∪	

local max

$f'(x) = 0 \Rightarrow x = 0$

$f''(x) = 0$ gives no real solutions

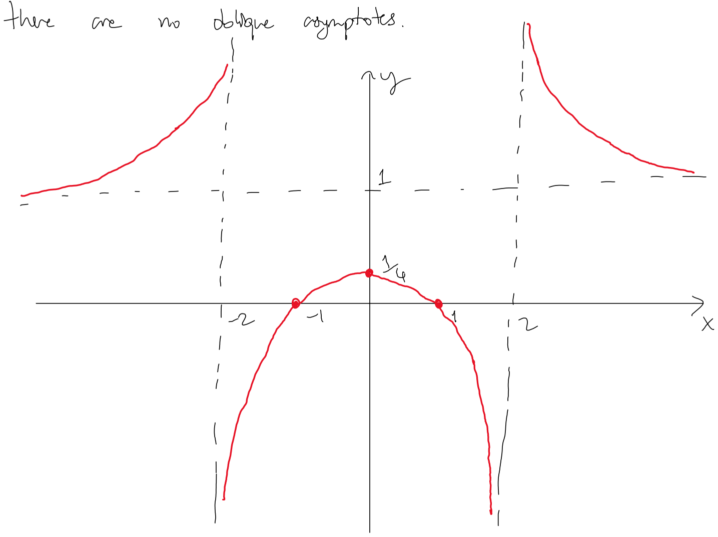
$\lim_{x\to 2^+} \frac{x^2-1}{(x-2)(x+2)} = +\infty$ and $\lim_{x\to 2^-} \frac{x^2-1}{(x-2)(x+2)} = -\infty$

$\lim_{x\to -2^+} \frac{x^2-1}{(x-2)(x+2)} = -\infty$ and $\lim_{x\to -2^-} \frac{x^2-1}{(x-2)(x+2)} = +\infty$

So we have two vertical asymptotes at $x = 2$ and $x = -2$.

$\lim_{x\to\infty} \frac{x^2-1}{x^2-4} = \lim_{x\to\infty} \frac{1 - \frac{1}{x^2}\to 0}{1 - \frac{4}{x^2}\to 0} = 1$ and $\lim_{x\to-\infty} \frac{x^2-1}{x^2-4} = 1$. So we have the horizontal asymptote $y = 1$.

Thus there are no oblique asymptotes.



$f(0) = \frac{0-1}{0-4} = \frac{1}{4}$

$f(x) = 0 \Rightarrow x = \mp 1$

Example: Sketch the graph of $f(x) = e^x \cdot (x^2-1)$.

Solution: We have $f'(x) = e^x \cdot (x^2-1) + e^x \cdot 2x = \underline{e^x(x^2+2x-1)}$ and

$f''(x) = e^x \cdot (x^2+2x-1) + e^x \cdot (2x+2) = \underline{e^x(x^2+4x+1)}$. So

$$f'(x) = 0 \Rightarrow x^2+2x-1 = 0 \Rightarrow x = \frac{-2 \mp \sqrt{4-4(-1)}}{2} = -1 \mp \sqrt{2}$$

$-1-\sqrt{2} \approx -2$
$-1+\sqrt{2} \approx 0$

$$f''(x) = 0 \Rightarrow x^2+4x+1 = 0 \Rightarrow x = \frac{-4 \mp \sqrt{16-4.1}}{2} = -2 \mp \sqrt{3}$$

$-2-\sqrt{3} \approx -$
$-2+\sqrt{3} \approx -$

x	$-\infty$		$-2-\sqrt{3}$		$-1-\sqrt{2}$		$-2+\sqrt{3}$		$-1+\sqrt{2}$		∞
f'		+		+	0	−		−	0	+	
f''		+	0	−		−	0	+		+	
f		↗ ∪		↗ ∩		↘ ∩		↘ ∪		↗ ∪	
			inflection point		local max		inflection point		local min		

There are no vertical asymptote[s]. f is continuous everywhere.

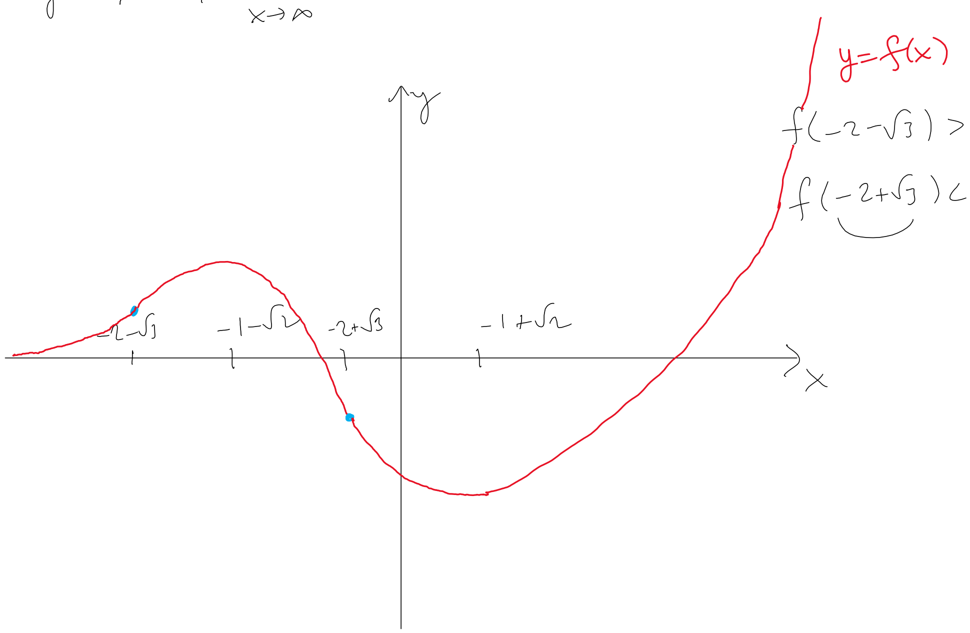
$\lim_{x\to\infty} e^x(x^2-1) = +\infty$ and

$$\lim_{x\to-\infty} e^x(x^2-1) = \lim_{x\to-\infty} \frac{x^2-}{\frac{1}{e}}$$

$$\lim_{x\to-\infty} \frac{2x}{\frac{-1}{(e^x)^2} \cancel{e^x}} \overset{L'H}{\underset{\left(\frac{+\infty}{+\infty}\right)}{=}} \lim_{x\to-\infty} \frac{2}{\frac{1}{(e^x)^2} \cancel{e^x}} = \lim_{x\to-\infty} 2e^x = 0$$

. So we have the hori[zontal] asymptote $y=0$

For any $m, n \in \mathbb{R}$, $\lim_{x\to\infty} \left(e^x(x^2-1) - (mx+n)\right) = +\infty$, so there are no oblique asy[mptotes]



$f(-2-\sqrt{3}) > 0$

$f(-2+\sqrt{3}) < 0$